Dynkin Game of Convertible Bonds and Their Optimal Strategy*

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Abstract

This paper studies the valuation and optimal strategy of convertible bonds as a Dynkin game by using the reflected backward stochastic differential equation method and the variational inequality method. We first reduce such a Dynkin game to an optimal stopping time problem with state constraint, and then in a Markovian setting, we investigate the optimal strategy by analyzing the properties of the corresponding free boundary, including its position, asymptotics, monotonicity and regularity. We identify situations when call precedes conversion, and vice versa. Moreover, we show that the irregular payoff results in the possibly non-monotonic conversion boundary. Surprisingly, the price of the convertible bond is not necessarily monotonic in time: it may even increase when time approaches maturity.

Key words: Convertible bond, Dynkin game, optimal stopping time problem, reflected BSDE, variational inequality, free boundary.

Mathematics subject classification: 35R35, 60H30, 91B25.

1 Introduction

Convertible bonds are often advertised as products with upside potential and limited downside risk, since a convertible bond is often supplemented with an option to exchange this bond for a given number of shares. The bondholder decides whether to keep the bond, in order to collect coupons, or to convert it to the firm's stocks. She will choose a conversion strategy to maximize the bond value. On the other hand, the issuing firm has the right to call the bond, and presumably acts to maximize the equity value of the firm by minimizing the bond value. This creates a *two-person, zero-sum Dynkin game*. One of the central questions for convertible bonds is to study such a Dynkin game, and more importantly, the corresponding optimal call and conversion strategies.

The study of convertible bonds dates back to Brennan and Schwartz [3] and Ingersoll [13]. However, both the call and conversion strategies are predetermined in these papers, so neither of them need to address the free boundary problem that arises if early conversion or early call is optimal. Sirbu et al

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[18] is one of the first to analyze the optimal strategy of perpetual convertible bonds (see also Sirbu and Shreve [19] for the finite horizon counterpart). They reduce the problem from a Dynkin game to an optimal stopping time problem, and discuss when call precedes conversion, and vice versa. Several more realistic features of convertible bonds have been taken into account since then. For example, Bielecki et al [1] consider the problem of the decomposition of a convertible bond into bond component and option component. Crépey and Rahal [5] study the convertible bond with call protection, which is typically path dependent, and more recently, Chen et al [4] consider the tax benefit and bankruptcy cost for convertible bonds. For a complete literature review, we refer to the aforementioned papers with references therein.

In this paper, we first study the Dynkin game of convertible bonds by using the reflected backward stochastic differential equation (*reflected BSDE for short*) method. Instead of regarding the convertible written on the stock value which is endogenously determined as the difference between the firm value and the bond value, we take a reduced form approach by assuming that the firm's stock value follows a general Itô process exogenously. Interestingly, similar to [18] and [19], we can also reduce the Dynkin game to an optimal stopping time problem with state constraint, i.e. reducing the reflected BSDE with two obstacles to a reflected BSDE with one obstacle and state constraint. An important consequence of this representation result is to allow us to identify when call precedes conversion, and vice versa, which is in line with [19]. That is, we show in Propositions 2.4-2.6 that when the coupon rate is bounded above by the interest rate times the surrender price, the bondholder will always convert her bond first; when the coupon rate is bounded below by the dividend rate times the surrender price, the firm will always call the bond first; when the coupon rate lies between the above two bounds, both the bondholder and the firm will terminate the contract simultaneously. We show that the above representation result holds in a general Itô process setting which is not necessarily Markovian, the latter of which is the standing assumption in both [18] and [19].

In the Markovian case, one way to study the optimal strategy of convertible bonds is to analyze the properties of the free boundary for the corresponding variational inequality (*VI for short*). Notwithstanding, the research on the free boundary analysis to understand the optimal strategy of convertible bonds is rare compared to the study on American options, for which the corresponding free boundary has already been well studied. One of the main reasons is that the corresponding Dynkin game (variational inequality) is too complicated to study. By utilizing the aforementioned representation result, we can reduce the corresponding Dynkin game to an optimal stopping time problem with state constraint, and this paves the way to study the properties of the corresponding free boundary. The current authors have already taken this path in some special cases (see [23, 25, 27]). For example, in [27] the authors assume that the issuer has no right to call. In [25] the authors only consider the surrender price and the final pre-specified price exactly equal, so the corresponding free boundary is always monotonic. In [23] only the case that the coupon rate is less than the interest rate times the surrender price is considered. In the present paper, we attempt to close the previous gaps, and give a complete analysis of the free boundary

under different cases, including the position of the free boundary with its asymptotics, monotonicity and regularity, etc. In particular, we concentrate on the case with irregular payoff (see Assumption 2.2).

There are several interesting properties of the free boundary as we prove in Section 3. First, it is well known that the asymptotics of the free boundary is more difficult to obtain than the asymptotics of the solution to the equation, because the convergence of the solution does not imply the convergence of the free boundary in general, and it is very difficult to deduce the latter via partial differential equation (*PDE for short*) estimates. In Theorem 3.3, we manage to obtain the asymptotics of both solution and free boundary. The main idea for the latter is as follows: we solve the corresponding perpetual problem, then use its solution to construct a sub-solution sequence and a super-solution sequence of the finite horizon problem, and show the asymptotic behavior of the free boundary via the two sequences.

Secondly, the free boundary in the VI (3.1) is non-monotonic under some parameter assumptions (see Theorem 3.4 and Figure 3.5). This is due to the singular terminal payoff which results in the blowup of the time derivative of the price near the maturity around the singular point. The non-monotonicity of the free boundary results in the non-monotonicity of the convertible bond price. In particular, the price may go up near maturity. In order to prove such a non-monotonicity property, we discuss its terminal asymptotic behavior and its initial asymptotic behavior as time goes to infinity, and prove that the terminal value is larger than the initial value, but less than the value at some middle point.

Thirdly, a standard assumption to prove the smoothness of the free boundary is that the difference between the solution and the lower obstacle of the VI is increasing with respect to time (see [10]). Without this monotonicity property, the regularity is difficult to achieve as discussed in [2, 17]. Unless the coupon rate is greater than the interest rate times the pre-specified price for the final payoff as in Theorem 3.5, this monotonicity condition does not hold, and the smoothness of the free boundary is not obvious at all. In Theorem 3.6 we show the smoothness of the free boundary even when this monotonicity condition fails, by using a subtle coordinate transformation and the comparison principle for VI.

The rest of the paper is organized as follows: In Section 2, we formulate our pricing model of convertible bonds as a Dynkin game by using the reflected BSDE method. In Section 3, we study the optimal strategies of convertible bonds by analyzing the properties of the corresponding free boundary. Some technical details about the solvability of the VI are presented in the appendix.

2 The Dynkin Game of Convertible Bonds

In this section, we formulate the pricing problem of convertible bonds as a zero-sum Dynkin game by using the reflected BSDE method. Our main result in this section is to show that such a Dynkin game can be reduced to an optimal stopping time problem with state constraint.

For a fixed time horizon $T > 0$, let W be a one dimensional Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}, \mathbb{P})$ satisfying the *usual conditions*, where $\mathbb{F}$ is the augmented filtration generated by the Brownian motion W, and $\mathbb{P}$ is interpreted as the risk-neutral probability measure. Consider a

firm who issues convertible bonds with the coupon rate c and the maturity T. The convertible bond is written on the firm's underlying stocks S, whose price process under the risk-neutral probability measure $\mathbb{P}$ is given by

$$S_s = S_t + \int_t^s (r_u - q_u) S_u \, du + \int_t^s \sigma_u S_u dW_u, \tag{2.1}$$

for $0 \le t \le s \le T$, where r, q, σ represent the risk-free interest rate, the dividend rate and the volatility, respectively.

Assumption 2.1 *The coupon rate c, the risk-free interest rate r, the dividend rate q and the volatility σ are $\mathbb{F}$-progressively measurable and uniformly bounded. Additionally, the volatility is positive $\sigma_t > 0$, a.s. for $t \in [0, T]$.*

Consider an investor purchasing a share of convertible bond from the issuer at any stating time $t \in [0, T]$. Assume there is no default for the firm. By holding the convertible bond, she will continuously receive the coupon rate c from the issuer until the contract is terminated. Prior to the contract maturity T, the investor has the right to convert her bond to the firm's stocks, while the firm has the right to call the bond and force the bondholder to surrender her bond to the firm. Hence there are three situations that the contract will be terminated: (1) if the firm calls the bond at some $\mathbb{F}$-stopping time τ first, the bondholder will receive a pre-specified surrender price K at time τ; (2) if the investor chooses to convert her bond at some $\mathbb{F}$-stopping time θ first or both players choose to stop the contract simultaneously, the bondholder will obtain γS_θ at time θ from converting her bond with a pre-specified conversion rate $\gamma \in (0, +\infty)$; (3) if neither players take any action during the contract period, then at the maturity T, the investor must sell her bond to the firm with a pre-specified price L or convert it to the firms' stocks with the conversion rate γ, so she will obtain $\max\{L, \gamma S_T\}$. In summary, the investor will obtain the following discounted payoff at the starting time $t \in [0, T]$:

$$\begin{aligned} P(\tau, \theta) \quad = \quad & \int_t^{\tau \wedge \theta} R(t, u) \, c_u \, du + R(t, \tau) \, K \, I_{\{\tau < \theta\}} + R(t, \theta) \, \gamma \, S_\theta \, I_{\{\theta \le \tau, \, \theta < T\}} \\ & + R(t, T) \, \max\{L, \, \gamma S_T\} \, I_{\{\tau \wedge \theta = T\}}, \end{aligned} \tag{2.2}$$

where $\tau, \theta \in \mathcal{U}_{t,T}$, the set of all $\mathbb{F}-$stopping times taking values in $[\, t, T\,]$, and $R(t, u) = \exp\{-\int_t^u r_s ds\}$ is the discount rate from t to u in the risk-neutral world.

The investor will choose $\theta \in \mathcal{U}_{t,T}$ to maximize $P(\tau, \theta)$, while the firm will choose $\tau \in \mathcal{U}_{t,T}$ to minimize $P(\tau, \theta)$. Hence we have the upper value and lower value, respectively,

$$\overline{V}_t = \operatorname*{ess.inf}_{\tau \in \mathcal{U}_{t,T}} \operatorname*{ess.sup}_{\theta \in \mathcal{U}_{t,T}} \mathbb{E}\,[\, P(\tau, \theta) | \mathcal{F}_t \,];$$

$$\underline{V}_t = \operatorname*{ess.sup}_{\theta \in \mathcal{U}_{t,T}} \operatorname*{ess.inf}_{\tau \in \mathcal{U}_{t,T}} \mathbb{E}\,[\, P(\tau, \theta) | \mathcal{F}_t \,]$$

of a corresponding Dynkin game (see [6] for the definition of Dynkin game). The value of this game exists if there exists some process V such that

$$V_t = \overline{V}_t = \underline{V}_t, \quad a.s. \text{ for } t \in [0, T],$$

and V_t is the time t value of this convertible bond by no-arbitrage principle (see Chapter 36 of [21]). It is standard to show that if there exits a Nash equilibrium point $(\tau_t^*, \theta_t^*) \in \mathcal{U}_{t,T} \times \mathcal{U}_{t,T}$ such that

$$\mathbb{E}\,[\,P(\tau_t^*, \theta)|\mathcal{F}_t\,] \le \mathbb{E}\,[\,P(\tau_t^*, \theta_t^*)|\mathcal{F}_t\,] \le \mathbb{E}\,[\,P(\tau, \theta_t^*)|\mathcal{F}_t\,], \quad a.s. \text{ for } \tau, \theta \in \mathcal{U}_{t,T},$$

then the value of this game V exists and is given by

$$V_t = \mathbb{E}\,[\,P(\tau_t^*, \theta_t^*)|\mathcal{F}_t\,]. \tag{2.3}$$

The Nash equilibrium point (τ_t^*, θ_t^*) is called the optimal strategy for such a convertible bond, where τ_t^* and θ_t^* represent the optimal calling and conversion strategy, respectively. The conversion payoff γS is usually called the lower obstacle, and the surrender price K called the upper obstacle.

Assumption 2.2 *The risk-free interest rate is no less than the dividend rate: $r_t \ge q_t \ge 0$, a.s. for $t \in [0, T]$, and the surrender price is greater than the maturity payment: $K > L > 0$.*

The first assumption $r_t \ge q_t \ge 0$ is natural. If $K < L$, then the pre-specified price L is irrelevant, since the firm could always call with the surrender price K before the maturity to avoid paying more (see [19]). If $K = L$, as shown in [27], the terminal value in the effective domain (state constraint) of the corresponding VI is just constant K, so the problem is relatively standard to study, and the corresponding free boundary is always monotonic. In this paper, we mainly consider the case $K > L$ which results in the singular terminal value across the free boundary, and this makes the problem much more complicated and involved. Moreover, the free boundary is not necessarily monotonic in this case.

In the following, we represent the optimal strategy (τ_t^*, θ_t^*) and the price V_t of the convertible bond in terms of the solution of reflected BSDE. Note that it is not always true that the conversion payoff (the lower obstacle) γS is dominated by the surrender price (the upper obstacle) K, so we have to resort to a reflected BSDE with the state constraint as follows.

Lemma 2.3 *Let (Y, Z, K^+, K^-) be the unique solution of the following reflected BSDE on $[t, \sigma_t^*]$:*

$$\begin{cases} Y_s = \max\{\gamma S_T, L\} I_{\{\sigma_t^* = T\}} + \gamma S_{\sigma_t^*} I_{\{\sigma_t^* < T\}} + \displaystyle\int_s^{\sigma_t^*} (c_u - r_u Y_u) du - \int_s^{\sigma_t^*} Z_u dW_u \\ \qquad + \displaystyle\int_s^{\sigma_t^*} dK_u^+ - \int_s^{\sigma_t^*} dK_u^-, \quad \gamma S_s \le Y_s \le K, \quad \mathit{for}\ s \in [\,t, \sigma_t^*\,], \\ \displaystyle\int_t^{\sigma_t^*} (Y_u - \gamma S_u) dK_u^+ = \int_t^{\sigma_t^*} (K - Y_u) dK_u^- = 0, \end{cases} \tag{2.4}$$

where

$$\sigma_t^* = \inf\{u \ge t : S_u \ge K/\gamma\} \wedge T.$$

Then the value of the convertible bond is given by $V_t = Y_t$ and the optimal strategy is given by

$$\tau_t^* = \inf\{s \ge t : Y_s = K\} \wedge \sigma_t^*, \qquad \theta_t^* = \inf\{s \ge t : Y_s = \gamma S_s\} \wedge \sigma_t^*.$$

The proofs of the above representation result and the well posedness of (2.4) are similar to Theorem 4.1 of Cvianic and Karatzas [6] with the fixed maturity T replaced by the random maturity σ_t^*, so we omit the proofs and refer to [6] for the details. Our main result in this section is to reduce (2.4) into an optimal stopping time problem with state constraint.

First note that if $S_t \geq K/\gamma$, i.e. the lower obstacle is greater than the upper obstacle, then $\sigma_t^* = t$, and in this case, both the investor and the firm will choose to terminate the contract at the same time $\tau_t^* = \theta_t^* = t$, and the value of the convertible bond is nothing but $V_t = \gamma S_t$. Hence, in the following we only consider the case $S_t < K/\gamma$.

Proposition 2.4 *Suppose that $c_s \leq r_s K$ a.s. on $s \in [t, \sigma_t^*]$. Then the value of the convertible bond is given by $V_t = Y_t^1$, where Y^1 solves the following reflected BSDE:*

$$\begin{cases} Y_s^1 = \max\{\gamma S_T, L\} I_{\{\sigma_t^* = T\}} + \gamma S_{\sigma_t^*} I_{\{\sigma_t^* < T\}} + \int_s^{\sigma_t^*} (c_u - r_u Y_u^1) du - \int_s^{\sigma_t^*} Z_u^1 dW_u \\ \qquad + \int_s^{\sigma_t^*} dK_u^{1,+}, \quad Y_s^1 \geq \gamma S_s, \quad \textit{for } s \in [\, t, \sigma_t^* \,], \\ \int_t^{\sigma_t^*} (Y_u^1 - \gamma S_u) dK_u^{1,+} = 0. \end{cases} \tag{2.5}$$

In particular, if $c_s < r_s K$ a.s. on $s \in [t, \sigma_t^*]$, then $Y_s^1 < K$ on $s \in [t, \sigma_t^*)$, so the optimal strategy is given by

$$\tau_t^* = \sigma_t^*, \qquad \theta_t^* = \inf\{s \geq t : Y_s^1 = \gamma S_s\} \wedge \sigma_t^*.$$

Proof. We first prove that $Y_s^1 \leq K$ on $s \in [t, \sigma_t^*]$. Then $(Y^1, Z^1, K^{1,+}, 0)$ is the solution to (2.4). Indeed, consider the following auxiliary reflected BSDE:

$$\begin{cases} \bar{Y}_s^1 = K + \int_s^{\sigma_t^*} (r_u K - r_u \bar{Y}_u^1) du - \int_s^{\sigma_t^*} \bar{Z}_u^1 dW_u + \int_s^{\sigma_t^*} d\bar{K}_u^{1,+}, \quad \bar{Y}_s^1 \geq \gamma S_s, \quad \text{for } s \in [\, t, \sigma_t^* \,], \\ \int_t^{\sigma_t^*} (\bar{Y}_u^1 - \gamma S_u) d\bar{K}_u^{1,+} = 0, \end{cases}$$

which obviously has a unique solution $(\bar{Y}_s^1, \bar{Z}_s^1, \bar{K}_s^{1,+}) = (K, 0, 0)$. Since

$$K \geq \max\{\gamma S_T, L\} I_{\{\sigma_t^* = T\}} + \gamma S_{\sigma_t^*} I_{\{\sigma_t^* < T\}},$$

and $r_s K - r_s Y_s^1 \geq c_s - r_s Y_s^1$ a.s. on $s \in [t, \sigma_t^*]$, the comparison principle of reflected BSDE (see [8]) implies that $Y_s^1 \leq \bar{Y}_s^1 = K$ on $s \in [t, \sigma_t^*]$.

Next we show that $Y_s^1 < K$ on $s \in [t, \sigma_t^*)$ if $c_s < r_s K$ a.s. on $s \in [t, \sigma_t^*]$. If not, there exits $\bar{s} \in [t, \sigma_t^*)$ such that $Y_{\bar{s}}^1 = K$. Note that we must have $Y_{\bar{s}}^1 > \gamma S_{\bar{s}}$ (otherwise $\gamma S_{\bar{s}} \geq Y_{\bar{s}}^1 = K$ would imply $\bar{s} = \sigma_t^*$). Define

$$\theta_{\bar{s}}^* = \inf\{s \geq \bar{s} : Y_s^1 = \gamma S_s\} \wedge \sigma_t^*.$$

Then $Y_{\theta_{\bar{s}}^*}^1 = \gamma S_{\theta_{\bar{s}}^*} \leq K$. Since $Y_s > \gamma S_s$ and $d\bar{K}_s^{1,+} = 0$ on $[\bar{s}, \theta_{\bar{s}}^*)$, (2.5) reads

$$Y_{\bar{s}}^1 = Y_{\theta_{\bar{s}}^*}^1 + \int_{\bar{s}}^{\theta_{\bar{s}}^*} (c_u - r_u Y_u^1) du - \int_{\bar{s}}^{\theta_{\bar{s}}^*} Z_u^1 dW_u.$$

Consider the following auxiliary BSDE:

$$\hat{Y}^1_{\bar{s}} = K + \int_{\bar{s}}^{\theta^*_{\bar{s}}} (r_u K - r_u \hat{Y}^1_u) du - \int_{\bar{s}}^{\theta^*_{\bar{s}}} \hat{Z}^1_u dW_u,$$

which obviously has a unique solution $(\hat{Y}^1_s, \hat{Z}^1_s) = (K, 0)$. Then the strict comparison principle of BSDE (see [7]) implies that $Y^1_{\bar{s}} < K$. ■

From the above proposition, if $c_s < r_s K$, the value of the convertible bond V_t is strictly less than the surrender price K before the termination of the contact, so the firm will not call the bond back until the contract is terminated at σ^*_t , and the investor will always convert her bond first.

Proposition 2.5 *Suppose that $c_s \geq q_s K$ a.s. on $s \in [t, \sigma^*_t]$. Then the value of the convertible bond is given by $V_t = Y^2_t$, where Y^2 solves the following reflected BSDE:*

$$\begin{cases} Y^2_s = \max\{\gamma S_T, L\} I_{\{\sigma^*_t = T\}} + \gamma S_{\sigma^*_t} I_{\{\sigma^*_t < T\}} + \displaystyle\int_s^{\sigma^*_t} (c_u - r_u Y^2_u) du - \int_s^{\sigma^*_t} Z^2_u dW_u \\ \qquad - \displaystyle\int_s^{\sigma^*_t} dK^{2,-}_u, \quad Y^2_s \leq K, \quad \textit{for } s \in [\, t, \sigma^*_t\,], \\ \displaystyle\int_t^{\sigma^*_t} (K - Y^2_u) dK^{2,-}_u = 0. \end{cases} \tag{2.6}$$

In particular, if $c_s > q_s K$ a.s. on $s \in [t, \sigma^*_t]$, then $Y^2_s > \gamma S_s$ on $s \in [t, \sigma^*_t)$, so the optimal strategy is given by

$$\tau^*_t = \inf\{s \geq t : Y^2_s = K\} \wedge \sigma^*_t, \qquad \theta^*_t = \sigma^*_t.$$

Proof. We first prove that $Y^2_s \geq \gamma S_s$ on $s \in [t, \sigma^*_t]$. Then $(Y^2, Z^2, 0, K^{2,-})$ is the solution to (2.4). Indeed, consider the following auxiliary reflected BSDE:

$$\begin{cases} \bar{Y}^2_s = \gamma S_{\sigma^*_t} + \displaystyle\int_s^{\sigma^*_t} (\gamma q_u S_u - r_u \bar{Y}^2_u) du - \int_s^{\sigma^*_t} \bar{Z}^2_u dW_u - \int_s^{\sigma^*_t} d\bar{K}^{2,-}_u, \quad \bar{Y}^2_s \leq K, \quad \text{for } s \in [\, t, \sigma^*_t\,], \\ \displaystyle\int_t^{\sigma^*_t} (K - \bar{Y}^2_u) d\bar{K}^{2,-}_u = 0, \end{cases}$$

which obviously has a unique solution $(\bar{Y}^2_s, \bar{Z}^2_s, \bar{K}^{2,-}_s) = (\gamma S_s, \gamma \sigma_s S_s, 0)$. Since

$$\gamma S_{\sigma^*_t} \leq \max\{\gamma S_T, L\} I_{\{\sigma^*_t = T\}} + \gamma S_{\sigma^*_t} I_{\{\sigma^*_t < T\}},$$

and $\gamma q_s S_s - r_s Y^2_s \leq q_s K - r_s Y^2_s \leq c_s - r_s Y^2_s$ on $s \in [t, \sigma^*_t]$, the comparison principle implies that $Y^2_s \geq \bar{Y}^2_s = \gamma S_s$ on $s \in [t, \sigma^*_t]$.

Next we show that $Y^2_s > \gamma S_s$ on $s \in [t, \sigma^*_t)$ if $c_s > q_s K$ a.s. on $s \in [t, \sigma^*_t]$. If not, there exits $\bar{s} \in [t, \sigma^*_t)$ such that $Y^2_{\bar{s}} = \gamma S_{\bar{s}}$. Note that we must have $Y^2_{\bar{s}} < K$ (otherwise $\gamma S_{\bar{s}} = Y^2_{\bar{s}} \geq K$ would imply that $\bar{s} = \sigma^*_t$). Define

$$\tau^*_{\bar{s}} = \inf\{s \geq \bar{s} : Y^2_s = K\} \wedge \sigma^*_t.$$

Then $Y^2_{\tau^*_{\bar{s}}} = K \geq \gamma S_{\tau^*_{\bar{s}}}$. Since $Y^2_s < K$, and $d\bar{K}^{2,-}_s = 0$ on $[\bar{s}, \tau^*_{\bar{s}})$, (2.6) reads

$$Y^2_{\bar{s}} = Y^2_{\tau^*_{\bar{s}}} + \int_{\bar{s}}^{\tau^*_{\bar{s}}} (c_u - r_u Y^2_u) du - \int_{\bar{s}}^{\tau^*_{\bar{s}}} Z^2_u dW_u.$$

Consider the following auxiliary BSDE:

$$\hat{Y}_{\bar{s}}^2 = \gamma S_{\tau_{\bar{s}}^*} + \int_{\bar{s}}^{\tau_{\bar{s}}^*} (\gamma q_u S_u - r_u \hat{Y}_u^2) du - \int_{\bar{s}}^{\tau_{\bar{s}}^*} \hat{Z}_u^2 dW_u,$$

which obviously has a unique solution $(\hat{Y}_s^2, \hat{Z}_s^2) = (\gamma S_s, \gamma \sigma_s S_s)$. Then the strict comparison principle implies that $Y_{\bar{s}}^1 > \gamma S_{\bar{s}}$. ■

From the above proposition, if $c_s > q_s K$, the value of the convertible bond V_t is strictly larger than the converting value γS_t before the termination of the contact, so the investor will not convert her bond until the contract is terminated at σ_t^* , and the firm will always call the bond first.

By repeating the arguments as in the proofs of Propositions 2.4 and 2.5, we obtain that the price can be represented as the solution of the following BSDE (2.7) if $q_s K \leq c_s \leq r_s K$. In particular, if $q_s K < c_s < r_s K$, then the value V_s of the convertible bond is bounded between $(\gamma S_s, K)$ before the termination of the contact. Hence, neither the investor will convert her bond nor the firm will call the bond back until the contract is terminated at σ_t^*.

Proposition 2.6 *Suppose that $q_s K \leq c_s \leq r_s K$ a.s. on $s \in [t, \sigma_t^*]$. Then the value of the convertible bond is given by $V_t = Y_t^3$, where Y^3 solves the following BSDE on $[t, \sigma_t^*]$:*

$$Y_s^3 = \max\{\gamma S_T, L\} I_{\{\sigma_t^* = T\}} + \gamma S_{\sigma_t^*} I_{\{\sigma_t^* < T\}} + \int_s^{\sigma_t^*} (c_u - r_u Y_u^3) du - \int_s^{\sigma_t^*} Z_u^3 dW_u. \tag{2.7}$$

In particular, if $q_s K < c_s < r_s K$ a.s. on $s \in [t, \sigma_t^*)$, then $Y_s^3 \in (\gamma S_s, K)$ on $s \in [t, \sigma_t^*)$, so the optimal strategy is given by

$$\tau_t^* = \theta_t^* = \sigma_t^*.$$

3 The Optimal Strategy of Convertible Bonds

In this section, we further consider the optimal strategy of convertible bonds in the Markovian case by investigating the properties of the corresponding calling/conversion boundaries.

Assumption 3.1 *Assume that all the coefficients are constants: $c_t = c, r_t = r > 0, q_t = q$, and $\sigma_t = \sigma$ for $t \in [0, T]$.*

Due to the above Markovian assumption, we know that there exists a function $V(S, t)$ such that $V_t = V(S_t, t)$. Define the following domains

$$\text{Conversion domain } \mathbf{CV} = \{(S,t) \in (0,\infty) \times [0,T) : V(S,t) = \gamma S\};$$

$$\text{Calling domain } \mathbf{CL} = \{(S,t) \in (0,\infty) \times [0,T) : V(S,t) = K \neq \gamma S\};$$

$$\text{Continuation domain } \mathbf{CT} = \{(S,t) \in (0,\infty) \times [0,T) : \gamma S < V(S,t) < K\}.$$

The intersecting line between the conversion domain **CV** and the continuation domain **CT** is called the conversion boundary $C(t)$, while the intersecting line between the calling domain **CL** and the continuation domain **CT** is called the calling boundary $H(t)$.

From the Feynman-Kac formula for the solution of reflected BSDE and the viscosity solution of VI (see Section 8 of [8]), Proposition 2.4 implies that if $c \le qK$ $(\le rK)$ then $V(S,t) = V^1(S,t)$ where V^1 solves the following VI with the state constraint:

$$\begin{cases} \partial_t V^1 + \mathcal{L}_0 V^1 = -c, & \text{if } V^1 > \gamma S \text{ and } (S,\,t) \in D_T, \\ \partial_t V^1 + \mathcal{L}_0 V^1 \le -c, & \text{if } V^1 = \gamma S \text{ and } (S,\,t) \in D_T, \\ V^1\,(K/\gamma, t) \;= K, & 0 \le t \le T, \\ V^1(S,T) \;= \max\{L, \gamma S\}, & 0 \le S \le K/\gamma, \end{cases} \tag{3.1}$$

where

$$\mathcal{L}_0 V^1 = \frac{\sigma^2}{2} S^2\,\partial_{SS} V^1 + (\,r - q\,)\,S\,\partial_S V^1 - rV^1, \quad D_T{=}(0, K/\gamma) \times [\,0, T).$$

Herein, D_T is the effective domain (the state constraint) of our problem, since in the domain $[K/\gamma, \infty) \times [\,0, T)$, $V(S,t) = \gamma S$, so the investor will always choose to convert, and $[K/\gamma, \infty) \times [0,T) \subset \mathbf{CV}$. Moreover, if $c < qK$, Proposition 2.4 also implies that $V^1(t, S) = Y^1_t < K$ on $(t, S) \in D_T$. Hence, we have $\mathbf{CL} \cap D_T = \emptyset$.

Similarly, Proposition 2.5 implies that if $c \ge rK$ $(\ge qK)$, then $V(S,t) = V^2(S,t)$ where V^2 solves the following VI with the state constraint:

$$\begin{cases} \partial_t V^2 + \mathcal{L}_0 V^2 = -c, & \text{if } V^2 < K \text{ and } (S,\,t) \in D_T, \\ \partial_t V^2 + \mathcal{L}_0 V^2 \ge -c, & \text{if } V^2 = K \text{ and } (S,\,t) \in D_T, \\ V^2\,(K/\gamma, t) \;= K, & 0 \le t \le T, \\ V^2(S,T) \;= \max\{L, \gamma S\}, & 0 \le S \le K/\gamma. \end{cases} \tag{3.2}$$

Moreover, if $c > rK$, Proposition 2.5 also implies that $V^2(t, S) = Y^2_t > \gamma S$ on $(t, S) \in D_T$, so $\mathbf{CV} \cap D_T = \emptyset$.

Finally, if $qK \le c \le rK$, then $V(S,t) = V^3(S,t)$ where V^3 solves the following Dirichlet problem:

$$\begin{cases} \partial_t V^3 + \mathcal{L}_0 V^3 = -c, & \text{in } D_T, \\ V^3\,(K/\gamma, t) \;= K, & 0 \le t \le T, \\ V^3(S,T) \;= \max\{L, \gamma S\}, & 0 \le S \le K/\gamma. \end{cases} \tag{3.3}$$

Moreover, the strong maximum principle (see [9]) implies that $V^3(t, S) = Y^3_t \in (\gamma S, K)$ on $(t, S) \in D_T$ (not only for the case $qK < c < rK$).

Therefore, the analysis of the calling/conversion strategies boils down to the properties of the free boundaries imbedded in the above three PDE problems.

The VI (3.2) for the case $c > rK$ has been studied in [23]. In such a case, the bondholder will not convert in the domain $(S,t) \in D_T$, and the calling boundary $H(t)$ is always monotonic (See Figure 3.1). The problem is therefore relatively standard. The PDE (3.3) for the case $qK \le c \le rK$ is trivial in the sense that neither the bondholder will convert nor the firm will call in the domain $(S,t) \in D_T$ (See Figure 3.2). We leave the explicit solution of the PDE (3.3) in Appendix B. In this paper, we mainly consider the VI (3.1) for the case $c < qK$. The situation in such a case is much more complicated and involved

(See Figure 3.3 – 3.5). The conversion boundary $C(t)$ may even lose the monotonicity property in such a case due to the singular payoff.

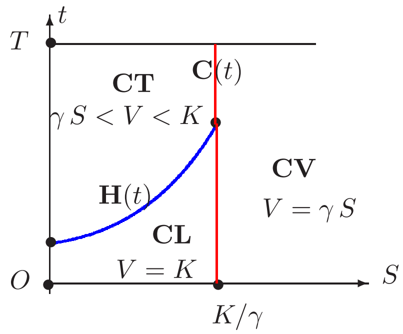

Figure 3.1. $c > rK$

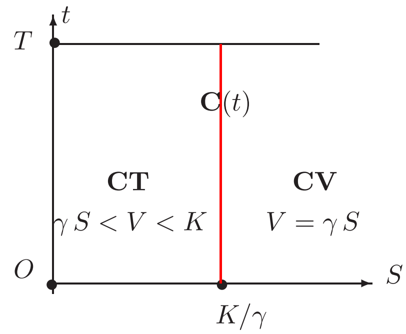

Figure 3.2. $qK \leq c \leq rK$

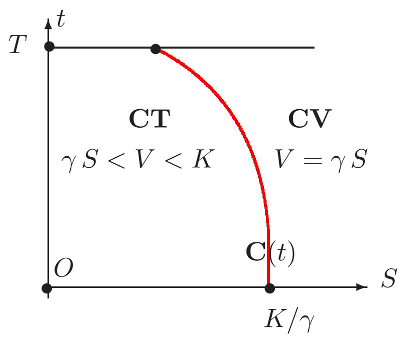

Figure 3.3. $rL \leq c < qK$, $c > \frac{(\alpha_+ - 1)\, rK}{\alpha_+}$

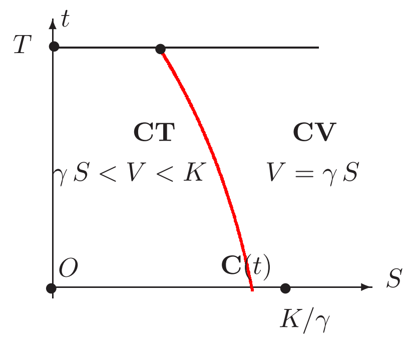

Figure 3.4. $rL \leq c < qK$, $c \leq \frac{(\alpha_+ - 1) rK}{\alpha_+}$

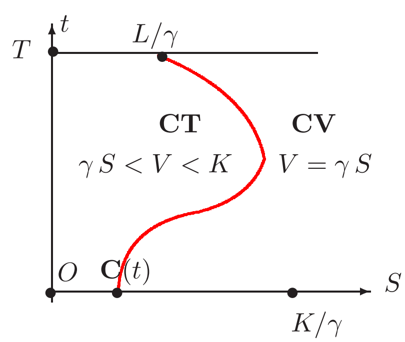

Figure 3.5. $c \leq rL(\alpha_+ - 1)/\alpha_+$

3.1 Properties of the Conversion Boundary

In this subsection, we prove the properties of the free boundary $C(t)$ of (3.1), such as its position, asymptotic property, monotonicity property, regularity property etc. We first show in Theorem 3.1 that the solution V^1 is not only the viscosity solution, but also the strong solution to (3.1): $V^1 \in W_p^{2,1}(D_T) \cap C(\overline{D_T})$ with $p > 1$.

Since (3.1) is degenerate, we first transform it into a familiar non-degenerate VI via the following transformation:

$$u(x, \tau) = V^1(S, t), \qquad \tau = T - t, \qquad x = \ln S - \ln K + \ln \gamma. \tag{3.4}$$

Then it is not difficult to check that u is governed by

$$\begin{cases} \partial_\tau u - \mathcal{L}u = c & \text{if } u > Ke^x \text{ and } (x,\tau) \in \Omega_T, \\ \partial_\tau u - \mathcal{L}u \geq c & \text{if } u = Ke^x \text{ and } (x,\tau) \in \Omega_T, \\ u(0,\tau) = K, & 0 \leq \tau \leq T, \\ u(x,0) = \max\{L, Ke^x\}, & x \leq 0, \end{cases} \tag{3.5}$$

where

$$\mathcal{L}u = \frac{\sigma^2}{2}\,\partial_{xx}u + \left(r - q - \frac{\sigma^2}{2}\right)\partial_x u - ru, \quad \Omega_T = (-\infty, 0) \times (0, T]. \tag{3.6}$$

Theorem 3.1 *For the case $c < qK$, the VI (3.5) has a unique strong solution $u \in W^{2,1}_{p,\,loc}(\Omega_T) \cap C(\overline{\Omega_T})$ with $p > 1$. Moreover, $\partial_x u \in C(\Omega_T)$ and we have the following estimates:*

$$\max\left\{Ke^x, \frac{c}{r} + \frac{rL - c}{r}\,e^{-r\tau}\right\} \leq u \leq K \quad \textit{in } \overline{\Omega}_T, \tag{3.7}$$

$$0 \leq \partial_x u \leq K\,e^x \qquad \textit{in } \overline{\Omega}_T. \tag{3.8}$$

If furthermore $c \geq rL$ holds, we also have the following estimate:

$$\partial_\tau u \geq 0 \ \ \textit{a.e. in } \Omega_T. \tag{3.9}$$

The proof of Theorem 3.1 is quite long and relatively standard, so we leave its proof in the appendix.

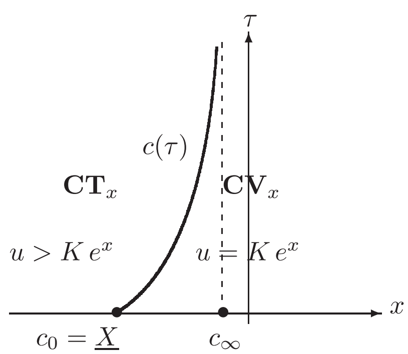

Figure 3.6. $c \geq rL,\ c \leq rK(\alpha_+ - 1)/\alpha_+$

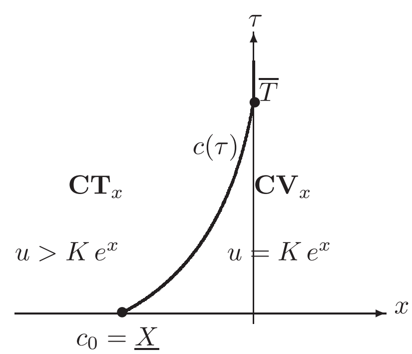

Figure 3.7. $c \geq rL,\ c > rK(\alpha_+ - 1)/\alpha_+$

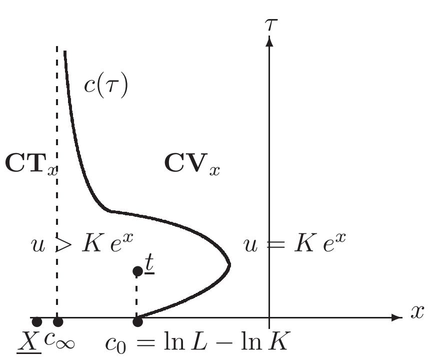

Figure 3.8. $c \leq rL(\alpha_+ - 1)/\alpha_+$

We denote $\mathbf{CV}_x$, $\mathbf{CT}_x$, $c(\tau)$ as the counterparts of $\mathbf{CV}$, $\mathbf{CT}$, $C(t)$ by the transformation (3.4), respectively. From (3.8), $u - Ke^x$ is decreasing with respect to x, so the conversion boundary is given as $c(\tau) = \inf\{x \le 0 : u(x,\tau) = Ke^x\}$, and the conversion region and the continuation region can further be characterized as

$$\mathbf{CV}_x = \{(x,\tau) \in (-\infty,0)\times(0,T] : u(x,\tau) = Ke^x\} = \{(x,\tau) \in (-\infty,0)\times(0,T] : x \ge c(\tau)\};$$
$$\mathbf{CT}_x = \{(x,\tau) \in (-\infty,0)\times(0,T] : u(x,\tau) > Ke^x\} = \{(x,\tau) \in (-\infty,0)\times(0,T] : x < c(\tau)\}.$$

Our main result in this section is to prove that the conversion and continuation regions have the following shapes under different parameter assumptions (see Figures 3.6 – 3.8). Note that Figure 3.8 shows that the conversion boundary is non-monotonic, and the price may go up when time approaches maturity around the starting point c_0 of the conversion boundary.

In the following, we prove the position, the asymptotics, the monotonicity and the regularity of the free boundary $c(\tau)$.

Theorem 3.2 *(Position of the free boundary)*

For the case $c < qK$, the free boundary $c(\tau)$ of the variational inequality (3.5) has the following properties:

(1) $\mathbf{CV}_x \subset \{x \ge \underline{X}\}$, so that $\mathbf{CT}_x \supset \{x < \underline{X}\}$ and $c(\tau) \in [\underline{X}, 0]$, where $\underline{X} \triangleq \ln c - \ln K - \ln q$.

(2) There exists a positive constant $\underline{t}$ such that $c(\tau) > c_0$ for any $\tau \in (0, \underline{t}]$ where $c_0 \triangleq \max\{\underline{X}, \ln L - \ln K\}$.

(3) The starting point of $c(\tau)$ is $(c(0), 0)$ with $c(0) \triangleq \lim\limits_{\tau\to 0^+} c(\tau) = c_0$.

Proof. (1). According to (3.5), in the domain $\mathbf{CV}_x \cap \Omega_T$, $V = Ke^x$ and it must hold

$$c \le \partial_\tau V - \mathcal{L}V = \partial_\tau(K\,e^x) - \mathcal{L}(K\,e^x) = q\,K\,e^x \Rightarrow x \ge \underline{X}.$$

Hence, $\mathbf{CV}_x \cap \Omega_T \subset \{x \ge \underline{X}\}$. Since $\mathbf{CL}_x \cap \Omega_T = \emptyset$, then $\mathbf{CT}_x \supset \{x < \underline{X}\}$ and $c(\tau) \ge \underline{X}$.

(2). The proof is divided into two cases:

<u>Case 1</u>: If $c_0 = \underline{X}$ (see Figures 3.6 and 3.7), then it is sufficient to prove that $c(\tau) > \underline{X}$ for any $\tau > 0$. Suppose not. Property (1) implies that there exists a $t_1 > 0$ such that $c(t_1) = \underline{X}$. We deduce that in the domain $\mathcal{N} = (-\infty, \underline{X}) \times (0, t_1]$: $u > Ke^x$, and u satisfies

$$\partial_\tau u - \mathcal{L}u = c \ge qKe^x = \partial_\tau(Ke^x) - \mathcal{L}(Ke^x), \quad u(\underline{X}, t_1) = Ke^x|_{x=\underline{X}}.$$

In view of the Hopf lemma (see [9]), we obtain that $\partial_x(u - Ke^x)(\underline{X}, t_1) < 0$. On the other hand, Theorem 3.1 implies that $\partial_x u \in C(\Omega_T)$. It means that $\partial_x u$ continuously crosses the free boundary $c(\tau)$, and $\partial_x(u - Ke^x)(\underline{X}, t_1) = 0$. Hence, we have a contradiction.

<u>Case 2</u>: If $c_0 = \ln L - \ln K$ (see Figure 3.8), then it is sufficient to prove that there exists a positive constant $\underline{t}$ such that

$$c(\tau) > \ln L - \ln K, \qquad \forall\ \tau \in (0, \underline{t}].$$

What we need to prove is that there exists a positive constant $\underline{t}$ such that

$$u(\ln L-\ln K,\tau)>K\,e^x|_{x=\ln L-\ln K}=L,\qquad \forall\ \tau\in(0,\underline{t}\,]. \tag{3.10}$$

Indeed, we denote w as the solution of PDE (B.1), then w takes the explicit form of (B.2). Hence, the A-B-P maximum principle (see [20]) implies that $u\geq w$ in $\overline{\Omega}_T$.

In order to prove (3.10), we use the explicit form of (B.2) to estimate asymptotic behavior of $w(\ln L-\ln K,\tau)$ as $\tau\to 0^+$. It is not difficult to check that as $\tau\to 0^+$, we have

$$\begin{cases}\text{if } x>0, & \Phi_1(x,\tau,t)=\Phi_2(x,\tau,t)=1+o(\sqrt{\tau}\,),\\ \text{if } x<0, & \Phi_1(x,\tau,t)=\Phi_2(x,\tau,t)=o(\sqrt{\tau}\,),\\ \text{if } x=0, & \Phi_1(x,\tau,t)=\frac{1}{2}-\frac{\sigma\,\alpha_1}{\sqrt{2\pi}}\sqrt{\tau-t}+o(\sqrt{\tau}\,),\\ \text{if } x=0, & \Phi_2(x,\tau,t)=\frac{1}{2}-\frac{\sigma\,(\alpha_1+1)}{\sqrt{2\pi}}\sqrt{\tau-t}+o(\sqrt{\tau}\,),\end{cases}$$

$$\begin{aligned} & w(\ln L-\ln K,\tau)-L\\ = {} & o(\sqrt{\tau}\,)+L\left(\frac{1}{2}-\frac{\sigma\,\alpha_1}{\sqrt{2\pi}}\sqrt{\tau}+o(\sqrt{\tau}\,)\right)-L\left(\frac{1}{2}-\frac{\sigma\,(\alpha_1+1)}{\sqrt{2\pi}}\sqrt{\tau}+o(\sqrt{\tau}\,)\right)\\ = {} & \frac{\sigma\,L}{\sqrt{2\pi}}\sqrt{\tau}+o(\sqrt{\tau}\,).\end{aligned}$$

Hence, there exists a positive constant $\underline{t}$ satisfies (3.10).

(3). Since we have proved the property (2), it is sufficient to show

$$\limsup_{\tau\to 0^+}c(\tau)\leq c_0.$$

The above inequality is obvious if we can prove that for any fixed $x_1>c_0$, there exists a positive constant δ^* such that

$$u(x_1,\tau)=K\,e^x|_{x=x_1},\quad \forall\,\tau\in[\,0,\delta^*\,]. \tag{3.11}$$

Indeed, for any fixed $x_1>c_0$, we construct a function such that

$$W(x,\tau)=Ke^x+\delta(x-x_1)^2,\quad (x,\tau)\in\mathcal{N}\triangleq[\,x_1-\delta,x_1+\delta\,]\times[\,0,\delta^*\,],$$

where δ, δ^* are positive constants to be determined. We first assume δ small enough so that $x_1-\delta>c_0$ and $x_1+\delta<0$. Next, we show that $u\leq W$ in $\overline{\mathcal{N}}$. Indeed, it is easy to check that in the domain $\mathcal{N}$, W satisfies

$$\begin{aligned}\partial_\tau W-\mathcal{L}W &= qKe^x-\delta\,[\sigma^2+(2r-2q-\sigma^2)(x-x_1)-r(x-x_1)^2]\\ &\geq qKe^{x_1-\delta}-\delta\,[\sigma^2+(2r+2q+\sigma^2)\delta\,]>c-\delta\,[\sigma^2+(2r+2q+\sigma^2)\delta\,],\end{aligned}$$

where we have used $x_1-\delta>c_0\geq\underline{X}$ in the last inequality. Choose δ small enough such that

$$\partial_\tau W-\mathcal{L}W>c\ \text{ in }\mathcal{N}.$$

Moreover, it is clear that

$$W(x_1 \pm \delta, 0) > Ke^x\Big|_{(x_1\pm\delta,0)} = u(x_1 \pm \delta, 0).$$

Recalling $u \in C(\overline{\Omega}_T\,)$, we deduce that there exists a positive constant δ^* such that

$$W(x_1 \pm \delta, \tau) \geq u(x_1 \pm \delta, \tau), \quad \forall\;\, \tau \in [\,0, \delta^*\,].$$

Hence, W satisfies

$$\begin{cases} \partial_\tau W - \mathcal{L}W \geq c, \quad W \geq Ke^x, \quad \text{in}\, \mathcal{N}, \\ W \geq u, \qquad \text{on}\; \partial_p \mathcal{N}. \end{cases}$$

The A-B-P maximum principle (see [20]) implies that $u \leq W$ in the domain $\overline{\mathcal{N}}$. In particularly, $u \leq W = Ke^x$ on the line $x = x_1$, $\tau \in (0, \delta^*\,]$. By combining $u \geq Ke^x$, we obtain (3.11). ■

Next, we analyze the asymptotic behavior of the free boundary and the solution of the VI (3.5) as $\tau \to \infty$:

Theorem 3.3 *(Asymptotics of the free boundary)*

For the case $c < qK$, the free boundary $c(\tau)$ and the solution $u(x, \tau)$ of the VI (3.5) has the following asymptotic properties:

(1) If furthermore $c \leq rK(\alpha_+ - 1)/\alpha_+$ holds, where α_+ is defined in Lemma A.1, then we have (see Figure 3.6 and Figure 3.8)

$$\lim_{\tau \to +\infty} c(\tau) = c_\infty \triangleq \ln\left(\frac{\alpha_+}{\alpha_+ - 1}\, \frac{c}{rK} \right),$$

$$\lim_{\tau \to +\infty} u(x, \tau) = u_{1,\infty}(x) \triangleq \begin{cases} \dfrac{K}{\alpha_+}\, \exp\Big\{\, \alpha_+ x + (1 - \alpha_+)\, c_\infty \Big\} + \dfrac{c}{r}, & x < c_\infty, \\ Ke^x, & c_\infty \leq x \leq 0. \end{cases}$$

(2) If furthermore $c > rK(\alpha_+ - 1)/\alpha_+$ holds, then there exists a positive constant $\overline{T}$ such that the free boundary $c(\tau)$ ends at the point $(0, \overline{T})$ (see Figure 3.7), i.e.,

$$c(\tau) = 0, \;\; \textit{for}\; \tau \in [\overline{T}, T], \quad \mathbf{CT}_x \supset (-\infty, 0) \times [\,\overline{T}, T\,],$$

$$\lim_{\tau \to +\infty} u(x, \tau) = u_{2,\infty}(x) \triangleq Ke^{\alpha_+ x} + \frac{c}{r}\, \left(\, 1 - e^{\alpha_+ x}\,\right), \quad x \leq 0.$$

Remark 3.1 *In fact, the above results imply that the solution $u(x, \tau)$ and the free boundary $c(\tau)$ of the finite horizon problem converge to the solution $u_{1,\infty}$ (or $u_{2,\infty}$) and the free boundary c_∞ (or 0) of the corresponding perpetual problem as time tends to infinity, respectively.*

Proof. The proof is divided into five steps:

Step 1: Construct a super-solution and a sub-solution of the VI (3.5).

For any fixed $t > 0$, we denote $\overline{u}_t$ as the $W^2_{p,\,loc}(\Omega) \cap C(\,\overline{\Omega}\,)$ solution of the following VI:

$$\begin{cases} -\mathcal{L}\, \overline{u}_t = c + re^{-rt/2}, & \text{if}\; \overline{u}_t > Ke^x \;\; \text{and} \;\; x \in \Omega \triangleq (-\infty, 0), \\ -\mathcal{L}\, \overline{u}_t \geq c + re^{-rt/2}, & \text{if}\; \overline{u}_t = Ke^x \;\; \text{and} \;\; x \in \Omega, \\ \overline{u}_t(0) \;= K. \end{cases} \tag{3.12}$$

We will give the explicit solution of the VI (3.12) in Step 2. Denote

$$W = \overline{u}_t + e^{rt/2 - r\tau} - e^{-rt/2}.$$

We claim that W is a super-solution of VI (3.5) if t is large enough. In fact, it is not difficult to check that

$$\begin{cases} \partial_\tau W - \mathcal{L}\, W \geq c \ \text{ and } \ W \geq Ke^x, \\ W(0,\tau) \geq K = u(0,\tau), \qquad\qquad 0 \leq \tau \leq t, \\ W(x,0) \geq K + e^{rt/2} - e^{-rt/2} \geq K \geq \max\{L, Ke^x\} = u(x,0), \quad x \leq 0. \ \text{(note that } r > 0) \end{cases}$$

By applying the comparison principle for VI (see [22]), we deduce that

$$u \leq W = \overline{u}_t + e^{rt/2 - r\tau} - e^{-rt/2}. \tag{3.13}$$

Next, denote $\underline{u}_{\,t}$ as the $W^2_{p,\,loc}(\Omega) \cap C(\,\overline{\Omega}\,)$ solution of the following VI:

$$\begin{cases} -\mathcal{L}\, \underline{u}_{\,t} = c - re^{-rt/2}, \qquad \text{if } \underline{u}_{\,t} > Ke^x \ \text{ and } \ x \in \Omega, \\ -\mathcal{L}\, \underline{u}_{\,t} \geq c - re^{-rt/2}, \qquad \text{if } \underline{u}_{\,t} = Ke^x \ \text{ and } \ x \in \Omega, \\ \underline{u}_{\,t}(0) \ = K. \end{cases} \tag{3.14}$$

We will give the explicit solution of the VI (3.14) in Step 2. Denote

$$w = \underline{u}_{\,t} - e^{rt/2 - r\tau} + e^{-rt/2}.$$

Repeating the same argument as above, we deduce that

$$u \geq w = \underline{u}_{\,t} - e^{rt/2 - r\tau} + e^{-rt/2}, \tag{3.15}$$

provided t is large enough.

<u>Step 2</u>: We solve the VIs (3.12) and (3.14). it is sufficient to solve the following elliptic VI:

$$\begin{cases} -\mathcal{L}\, v = c^*, \qquad \text{if } v > Ke^x \ \text{ and } \ x \in \Omega, \\ -\mathcal{L}\, v \geq c^*, \qquad \text{if } v = Ke^x \ \text{ and } \ x \in \Omega, \\ v(0) \ = K. \end{cases} \tag{3.16}$$

It is clear that (3.12) and (3.14) coincide with the VI (3.16) if we let $c^* = c + re^{-rt/2}$ and $c^* = c - re^{-rt/2}$, respectively.

(1) In the case $c^* \leq rK(\alpha_+ - 1)/\alpha_+$, we first find out the bounded solution of the following associated free boundary problem of (3.16):

$$\begin{cases} -\mathcal{L}\, v = c^* > 0, \quad x \in (-\infty, x^*), \\ \partial_x v(x^*) = v(x^*) = Ke^{x^*}. \end{cases} \tag{3.17}$$

It is not difficult to check that the solution of (3.17) should take the form of

$$v = A\, e^{\alpha_+ x} + B\, e^{\alpha_- x} + \frac{c^*}{r}, \qquad x < x^*,$$

where α_- is defined in Lemma A.1. Since v is bounded and $\alpha_- < 0$, then we have $B = 0$. Recalling the boundary condition, we deduce

$$A\,e^{\alpha_+ x^*} = Ke^{x^*} - \frac{c^*}{r}, \qquad A\,\alpha_+\,e^{\alpha_+ x^*} = Ke^{x^*}.$$

Since $\alpha_+ > 1$, then we have

$$x^* = \ln\left(\frac{\alpha_+}{\alpha_+ - 1}\,\frac{c^*}{rK}\right), \qquad v = \frac{K}{\alpha_+}\,e^{\alpha_+ x + (1-\alpha_+)\,x^*} + \frac{c^*}{r}. \tag{3.18}$$

It is clear that $x^* \le 0$. Extend v into $(-\infty, 0\,]$ as follows:

$$v(x) = \begin{cases} \dfrac{K}{\alpha_+}\,\exp\{\,\alpha_+ x + (1-\alpha_+)\,x^*\,\} + \dfrac{c^*}{r}, & x < x^*, \\ Ke^x, & x^* \le x \le 0. \end{cases} \tag{3.19}$$

Next, we prove that v is the unique $W^2_{p,\,loc}(\Omega) \cap C(\,\overline{\Omega}\,) \cap L^\infty(\Omega)$ solution of the VI (3.16). In fact, the uniqueness follows from the comparison principle for VI (see [22]), and it is easy to verify the regularity of the solution. Then it is sufficient to prove that v satisfies the VI (3.16). According to (3.18), we can check that

$$\partial_x v(x) = K\,e^{\alpha_+ x + (1-\alpha_+)\,x^*} = Ke^x\,e^{(\alpha_+ - 1)(x - x^*)} \le Ke^x = \partial_x(Ke^x), \quad x \le x^*.$$

By combining the boundary condition of (3.17), we obtain that

$$v(x) - Ke^x \ge 0,\ \forall\, x \le x^*.$$

Hence, we only need to prove that

$$c^* \le -\mathcal{L}\,Ke^x = q\,Ke^x,\ \ \forall\ \ x > x^*.$$

It is sufficient to show that

$$c^* \le q\,Ke^{x^*} = q\,K\,\frac{\alpha_+}{\alpha_+ - 1}\,\frac{c^*}{rK} \Leftrightarrow \frac{\alpha_+}{\alpha_+ - 1} \ge \frac{r}{q} \Leftrightarrow \alpha_+ \le \frac{r}{r-q}. \tag{3.20}$$

In fact, it is easy to check that

$$\frac{\sigma^2}{2}\left(\frac{r}{r-q}\right)^2 + \left(r - q - \frac{\sigma^2}{2}\right)\left(\frac{r}{r-q}\right) - r = \frac{\sigma^2}{2}\left[\left(\frac{r}{r-q}\right)^2 - \frac{r}{r-q}\right] > 0.$$

Recalling the definition of α_+, we deduce (3.20) from the property of quadratic functions. Hence, we have checked that v is the uniqueness solution of the VI (3.16).

(2) In the case of $c^* > rK(\alpha_+ - 1)/\alpha_+$, since x^* defined in (3.18) is larger than zero, then v defined in (3.19) is not the solution of the VI (3.16). Now, we need to reconstruct the solution of the VI (3.16). We first solve the following ODE

$$-\,\mathcal{L}\,v = c^* > 0,\ \ x \in (-\infty, 0); \qquad v(0) = K. \tag{3.21}$$

It is not difficult to check that the bounded solution is

$$v(x) = Ke^{\alpha_+ x} + \frac{c^*}{r}\left(1 - e^{\alpha_+ x}\right), \quad x \le 0. \tag{3.22}$$

Next, we prove that v is the unique $W^2_{p,\,loc}(\Omega)\cap C(\overline{\Omega})\cap L^\infty(\Omega)$ solution of the VI (3.16). By the same argument as above, it is sufficient to prove $v(x) \ge Ke^x$ for any $x \le 0$. Indeed, we calculate

$$\partial_x v(x) = \alpha_+\left(K - \frac{c^*}{r}\right)e^{\alpha_+ x} \le Ke^{\alpha_+ x} \le Ke^x, \ \ \forall\, x \le 0,$$

where we have used $\alpha_+ > 1$. By combining the boundary condition of (3.21), we deduce that $v(x) \ge Ke^x$ for any $x \le 0$. Hence, we have showed that v is the unique solution of the VI (3.16).

Step 3: Prove the property (1) in the case of $c < rK(\alpha_+ - 1)/\alpha_+$.

In view of (3.13) and (3.15), we deduce the following inequality if t is large enough,

$$\underline{u}_{\,t}(x) - e^{rt/2 - r\tau} + e^{-rt/2} \le u(x,\tau) \le \overline{u}_t(x) + e^{rt/2 - r\tau} - e^{-rt/2}.$$

In particular, by taking $\tau = t$ we have

$$\underline{u}_{\,t}(x) \le u(x,t) \le \overline{u}_t(x).$$

Since $\underline{u}_{\,t}$, u, $\overline{u}_t \ge Ke^x$, we derive

$$\{x : \underline{u}_{\,t}(x) = Ke^x\} \supset \{x : u(x,t) = Ke^x\} \supset \{x : \overline{u}_t(x) = Ke^x\}.$$

It is not difficult to check that

$$c + re^{-rt/2},\, c - re^{-rt/2} \to c < rK(\alpha_+ - 1)/\alpha_+ \ \text{ as } t \to +\infty.$$

Hence, the conclusion in Step 2 implies that $\underline{u}_{\,t}$, $\overline{u}_t$ takes the form of (3.19) with $c^* = c - re^{-rt/2}$ and $c^* = c + re^{-rt/2}$, respectively. Denote $\underline{x}_{\,t}$, $\overline{x}_t$ as the corresponding free boundary points x^* defined in (3.18). Since t is arbitrary, then we have

$$[\,\underline{x}_{\,\tau}, 0\,] = \{x : \underline{u}_{\,\tau}(x) = Ke^x\} \supset \{x : u(x,\tau) = Ke^x\} \supset \{x : \overline{u}_\tau(x) = Ke^x\} = [\,\overline{x}_\tau, 0\,],$$

provided τ is large enough. Hence, the definition of the free boundary $c(\tau)$ implies that

$$\ln\left(\frac{\alpha_+}{\alpha_+ - 1}\,\frac{c - re^{-r\tau/2}}{rK}\right) = \underline{x}_{\,\tau} \le c(\tau) \le \overline{x}_\tau = \ln\left(\frac{\alpha_+}{\alpha_+ - 1}\,\frac{c + re^{-r\tau/2}}{rK}\right) < 0,$$

provided τ is large enough. Moreover, it is not difficult to check that

$$\lim_{\tau\to+\infty}\underline{x}_{\,\tau} = c_\infty = \lim_{\tau\to+\infty}\overline{x}_\tau, \quad \lim_{\tau\to+\infty}\underline{u}_{\,\tau}(x) = u_{1,\infty}(x) = \lim_{\tau\to+\infty}\overline{u}_{\,\tau}(x),\ \forall\, x \in \overline{\Omega}.$$

Hence, the property (1) follows.

Step 4: Prove the property (1) in the case of $c = rK(\alpha_+ - 1)/\alpha_+$.

In this case, $c - re^{-rt/2} < rK(\alpha_+ - 1)/\alpha_+$, and $\underline{u}_{\,t}$ still takes the form of (3.19) if t is large enough. Repeating same the argument as in Step 3, we still have that

$$c(\tau) \geq \underline{x}_{\,\tau} = \ln\left(\frac{\alpha_+}{\alpha_+ - 1}\,\frac{c - re^{-r\tau/2}}{rK}\right), \qquad \liminf_{\tau\to+\infty} c(\tau) \geq \lim_{\tau\to+\infty} \underline{x}_{\,\tau} = 0 = c_\infty,$$

$$\liminf_{\tau\to+\infty} u(x,\tau) \geq \lim_{\tau\to+\infty} \underline{u}_{\,\tau}(x) = u_{1,\infty}(x) = \frac{K}{\alpha_+}\, e^{\alpha_+ x} + \frac{c}{r}, \quad \forall\, x \in \overline{\Omega},$$

provided τ is large enough.

On the other hand, the definition of the free boundary $c(\tau)$ implies that $c(\tau) \leq 0$. Hence, we deduce that

$$\lim_{\tau\to+\infty} c(\tau) = 0 = c_\infty.$$

By applying the same method as in Step 3, we derive that

$$\limsup_{\tau\to+\infty} u(x,\tau) \leq \lim_{\tau\to+\infty} \overline{u}_\tau(x), \quad \forall\, x \in \overline{\Omega}.$$

Since $c + re^{-r\tau/2} > (\alpha_+ - 1)\, rK/\alpha_+$, $\overline{u}_\tau$ takes form of (3.22) rather than (3.19). It is easy to calculate that

$$\lim_{\tau\to+\infty} \overline{u}_\tau = \left(K - \frac{c}{r}\right) e^{\alpha_+ x} + \frac{c}{r} = \left(K - \frac{(\alpha_+ - 1)\, rK}{\alpha_+ r}\right) e^{\alpha_+ x} + \frac{c}{r} = u_{1,\infty}(x).$$

From the above arguments, we have that

$$\liminf_{\tau\to+\infty} u(x,\tau) \geq \lim_{\tau\to+\infty} \underline{u}_{\,\tau}(x) = u_{1,\infty}(x) = \lim_{\tau\to+\infty} \overline{u}_\tau(x) \geq \limsup_{\tau\to+\infty} u(x,\tau).$$

Hence, we have proved the property (1) in the case of $c = rK(\alpha_+ - 1)/\alpha_+$.

Step 5: Prove the property (2).

In this case, $\underline{u}_{\,\tau}$, $\overline{u}_\tau$ take the form of (3.22) if τ is large enough. Repeating the same arguments as in Step 3, we get

$$\{x = 0\} = \{x : \underline{u}_{\,\tau}(x) = Ke^x\} \supset \{x : u(x,\tau) = Ke^x\} \supset \{x : \overline{u}_\tau(x) = Ke^x\} = \{x = 0\},$$

provided τ is large enough. Then the definition of the free boundary $c(\tau)$ implies that $c(\tau) = 0$ if τ is large enough. Hence, there exists a positive constant $\overline{T}$ such that

$$c(\tau) = 0, \;\; \forall\, \tau \geq \overline{T}.$$

It is clear that

$$\lim_{\tau\to+\infty} \underline{u}_{\,\tau}(x) = u_{2,\infty}(x) = \lim_{\tau\to+\infty} \overline{u}_\tau(x), \;\; \forall\, x \in \overline{\Omega}.$$

Hence, the property (2) follows. ■

In view of the properties (2), (3) in Theorem 3.2 and the property (1) in Theorem 3.3, we claim the non-monotonicity property of the free boundary $c(\tau)$ (see Figure 3.8).

Theorem 3.4 *(Non-monotonicity of the free boundary)*

For the case $c < qK$, if furthermore $c \leq rL(\alpha_+ - 1)/\alpha_+$ holds, then the free boundary $c(\tau)$ is non-monotonic in the interval $[\,0, T\,]$ (where we suppose that T is large enough).

Proof. If $c_0 \geq c_\infty$, then the properties (2), (3) in Theorem 3.2 imply that there exists a $t_1 > 0$ such that

$$c(t_1) > c_0 = c(0) \geq c_\infty.$$

According to the property (1) in Theorem 3.3, we know that there exists a t_2 large enough such that $t_2 > t_1$ and

$$c(t_2) \leq \frac{c_\infty + c(t_1)}{2} < c(t_1).$$

Hence, the free boundary $c(\tau)$ is non-monotonic. On the other hand, it is clear that

$$c_0 \geq c_\infty \Leftrightarrow \max\left\{\frac{c}{qK}, \frac{L}{K}\right\} \geq \frac{\alpha_+}{\alpha_+ - 1}\,\frac{c}{rK} \Leftrightarrow \max\left\{\frac{c}{q}, L\right\} \geq \frac{\alpha_+}{\alpha_+ - 1}\,\frac{c}{r}.$$

By applying the same method as in the proof of (3.20), we conclude that

$$\alpha_+ < \frac{r}{r-q} \Leftrightarrow (r-q)\alpha_+ < r \Leftrightarrow \frac{c}{q} < \frac{\alpha_+}{\alpha_+ - 1}\,\frac{c}{r}.$$

Hence,

$$c_0 \geq c_\infty \Leftrightarrow L \geq \frac{\alpha_+}{\alpha_+ - 1}\,\frac{c}{r} \Leftrightarrow c \leq \frac{rL\,(\alpha_+ - 1)}{\alpha_+}.$$

■

Next, we consider the monotonicity and regularity of the free boundary $c(\tau)$ if $c \geq rL$. Since (3.9) holds, the problem is relatively standard in this case.

Theorem 3.5 *(Regularity of the free boundary) For the case $c < qK$, if furthermore $c \geq rL$ holds, the free boundary $c(\tau)$ is increasing with respect to τ on the interval $[\,0,T\,]$ with $c(\tau) \in C[\,0,T\,] \cap C^\infty(0,T\,]$. Moreover, $c(\tau)$ is strictly increasing on $[\,0,\underline{T}\,]$ with $\underline{T} = \sup\{\tau \in [\,0,T\,] : c(\tau) < 0\}$.*

Proof. According to (3.8) and (3.9), we have

$$\partial_x(u - Ke^x) \leq 0, \;\; \partial_\tau(u - Ke^x) \geq 0 \;\text{ a.e. in } \Omega_T.$$

By combining $u - Ke^x \in C(\,\overline{\Omega_T}\,)$, we deduce that $u(x,\tau) - Ke^x$ is increasing with respect to τ and decreasing with respect to x.

For any fixed $\tau_0 \in (0,T\,]$ and any $x \in [\,c(\tau_0), 0\,]$, $\tau \in [\,0,\tau_0\,]$, we derive that

$$0 \leq u(x,\tau) - Ke^x \leq u(c(\tau_0),\tau) - Ke^{c(\tau_0)} \leq u(c(\tau_0),\tau_0) - Ke^{c(\tau_0)} = 0,$$

where we have used that $u = Ke^x$ on the free boundary. Hence, the definition of the free boundary implies that $c(\tau) \leq c(\tau_0)$ for any $\tau \in [\,0,\tau_0\,]$. Hence, we deduce that $c(\tau)$ is increasing on $[\,0,T\,]$.

The property (3) in Theorem 3.2 implies that $c(\tau)$ is right-continuous at $\tau = 0$. Next, we prove that $c(\tau)$ is continuous on $(0,T\,]$. Otherwise, there exist some constants x_1, x_2, t_1 such that $x_2 < x_1 \leq 0$, $0 < t_1 < T$, $\lim_{\tau\to t_1^-} c(\tau) = x_2$, $\lim_{\tau\to t_1^+} c(\tau) = x_1$ (see Figure 3.9), and

$$\partial_\tau u - \mathcal{L}u = c \;\text{ in }\; (x_2,x_1) \times [\,t_1,T\,], \qquad u(x,t_1) = Ke^x, \;\forall\, x \in (x_2,x_1).$$

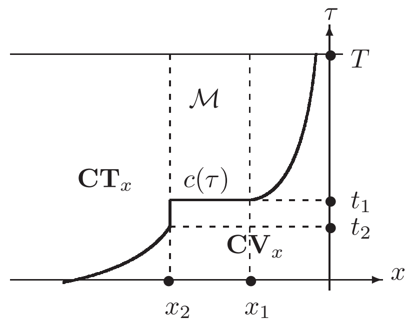

Figure 3.9. Non-continuous free boundary

If $x_2 < x < x_1$, then we have

$$\partial_\tau u(x, t_1) = c + \mathcal{L} K e^x = c - qK\, e^x < 0,$$

where the last inequality follows from $x_2 \geq \underline{X}$, which is deduced from the property (1) in Theorem 3.2. It is clear that the above inequality contradicts (3.9).

Next, we prove that $c(\tau)$ is strictly increasing on $[\,0, \underline{T}\,]$. Otherwise, there exist some constants $x_2,\, t_1,\, t_2$ such that $x_2 < 0,\; 0 \leq t_2 < t_1 \leq \underline{T}$ and $c(\tau) = x_2$ for any $\tau \in [\,t_2, t_1\,]$ (see Figure 3.9). It is clear that $u(x, \tau) = Ke^x$ for any $(x, \tau) \in [\,x_2, 0\,] \times [\,t_2, t_1\,]$. Since $\partial_x u$ continuously crosses the free boundary, then $\partial_x u(x_2, \tau) = Ke^{x_2}$ for any $\tau \in [\,t_2, t_1\,]$. We then deduce that

$$\partial_\tau u(x_2, \tau) = 0, \;\; \partial_\tau(\partial_x u)(x_2, \tau) = 0, \quad \forall\, \tau \in [\,t_2, t_1\,]. \tag{3.23}$$

On the other hand, in the domain $\mathcal{N} = (-\infty, x_2) \times (t_2, t_1\,]$, u and $\partial_\tau u$ respectively satisfies

$$\partial_\tau u - \mathcal{L} u = c \;\text{ in }\; \mathcal{N}, \qquad u(x_2, \tau) = Ke^{x_2},\; \forall\, \tau \in (t_2, t_1),$$

$$\begin{cases} \partial_\tau(\partial_\tau u) - \mathcal{L}(\partial_\tau u) = 0, \quad \partial_\tau u \geq 0 \;\text{ in }\; \mathcal{N}, \\ \partial_\tau u(x_2, \tau) = 0,\; \forall\, \tau \in (t_2, t_1). \end{cases}$$

By applying the Hopf lemma, we deduce $\partial_x(\partial_\tau u)(x_2, \tau) < 0$, which contradicts the second equality in (3.23).

Finally, since we have the estimate (3.9), it is standard to show that $C^\infty(0, T\,]$ (see [10]). ■

Next, we improve the regularity of the free boundary $c(\tau)$ for the case $c < rL$. In this case, (3.9) is false, so the standard method in [10] does not apply to this problem. The main idea to improve the regularity is to apply some proper coordinate transformation to the original problem, and transform it into a new problem, and achieve the estimate similar to (3.9).

Theorem 3.6 *(Regularity of the free boundary)*

For the case $c < qK$, if furthermore $c < rL$ holds, then the free boundary $c(\tau) \in C[\,0, T\,] \cap C^\infty(0, T\,]$.

Proof. We first apply the following transformation

$$y = x + \left(r - \frac{c}{L}\right)\tau, \qquad v(y, \tau) = e^{(r - \frac{c}{L})\,\tau}\,(\,u(x, \tau) - Ke^x\,). \tag{3.24}$$

It is not difficult to deduce that v satisfies the following VI (see Figure 3.10):

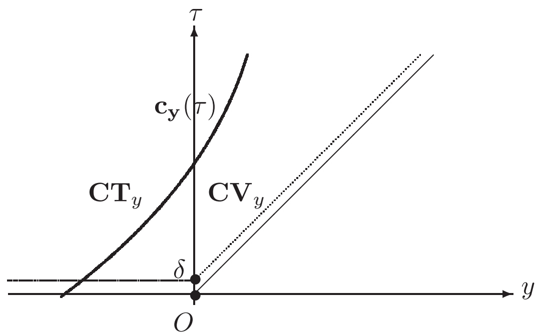

Figure 3.10. The free boundary $c_y(\tau)$ after transformation

$$\begin{cases} \partial_\tau v - \mathcal{L}_y v = ce^{(r-\frac{c}{L})\tau} - qKe^y, & \text{if } v > 0 \text{ and } (y,\tau) \in \Omega_T^y, \\ \partial_\tau v - \mathcal{L}_y v \geq ce^{(r-\frac{c}{L})\tau} - qKe^y, & \text{if } v = 0 \text{ and } (y,\tau) \in \Omega_T^y, \\ v((r-c/L)\tau, \tau) = 0, & 0 \leq \tau \leq T, \\ v(y,0) = (L - Ke^y)^+, & y \leq 0, \end{cases} \tag{3.25}$$

where

$$\mathcal{L}_y v = \frac{\sigma^2}{2}\partial_{yy} v + \left(\frac{c}{L} - q - \frac{\sigma^2}{2}\right)\partial_y v - \frac{c}{L} v, \ \ \Omega_T^y \triangleq \left\{ y < \left(r - \frac{c}{L}\right)\tau, \ 0 < \tau \leq T \right\}.$$

For any small enough $\delta > 0$, we denote

$$\widetilde{v}(y,\tau) = v(y, \tau+\delta), \ \ (y,\tau) \in \Omega_{T-\delta}^y.$$

Then $\widetilde{v}$ satisfies the following VI (see Figure 3.10):

$$\begin{cases} \partial_\tau \widetilde{v} - \mathcal{L}_y \widetilde{v} = ce^{(r-\frac{c}{L})(\tau+\delta)} - qKe^y \geq ce^{(r-\frac{c}{L})\tau} - qKe^y, & \text{if } \widetilde{v} > 0 \text{ and } (y,\tau) \in \Omega_{T-\delta}^y, \\ \partial_\tau \widetilde{v} - \mathcal{L}_y \widetilde{v} \geq ce^{(r-\frac{c}{L})(\tau+\delta)} - qKe^y \geq ce^{(r-\frac{c}{L})\tau} - qKe^y, & \text{if } \widetilde{v} = 0 \text{ and } (y,\tau) \in \Omega_{T-\delta}^y, \\ \widetilde{v}((r-c/L)\tau, \tau) = v((r-c/L)\tau, \tau+\delta) \geq 0, & 0 \leq \tau \leq T-\delta, \\ \widetilde{v}(y,0) = v(y,\delta), & y \leq 0. \end{cases}$$

Next, we prove $\widetilde{v} \geq v$ in $\overline{\Omega_{T-\delta}^y}$. In fact, the comparison principle for VI (see [22]) implies that it is sufficient to show that

$$\widetilde{v}(y,0) = v(y,\delta) \geq (L - Ke^y)^+ = v(y,0).$$

Moreover, since $v \geq 0$, then what we need to prove is that $L - Ke^y$ is a subsolution of (3.25). Indeed, we can check that

$$\begin{cases} \partial_\tau (L - Ke^y) - \mathcal{L}_y (L - Ke^y) = c - qKe^y \leq ce^{(r-\frac{c}{L})\tau} - qKe^y, \\ (L - Ke^y)\Big|_{y=(r-c/L)\tau} \leq 0 = v(y,\tau)\Big|_{y=(r-c/L)\tau}, \ \ 0 \leq \tau \leq T. \end{cases}$$

Hence, we conclude that $L - Ke^y$ is indeed a subsolution of (3.25).

We have showed $v(y,\tau+\delta) = \widetilde{v}(y,\tau) \geq v(y,\tau)$ in $\overline{\Omega_{T-\delta}^y}$ for any small enough δ, which implies $\partial_\tau v \geq 0$ almost everywhere in Ω_T^y. Hence, by using the method as in [10], we can prove that $c_y(\tau) \in C[0,T] \cap C^\infty(0,T]$. According to the transformation (3.24), we have $c_x(\tau) = c_y(\tau) - \left(r - \frac{c}{L}\right)\tau$. Therefore, $c(\tau) \in C[0,T] \cap C^\infty(0,T]$. ∎

A The Proof of Theorem 3.1

We prove Theorem 3.1 in this appendix. Since (3.5) lies in the unbounded domain Ω_T, we use the following VI in the bounded domain to approximate (3.5),

$$\begin{cases} \partial_\tau u_n - \mathcal{L}u_n = c, & \text{if } u_n > Ke^x \text{ and } (x,\tau) \in \Omega_T^n, \\ \partial_\tau u_n - \mathcal{L}u_n \geq c, & \text{if } u_n = Ke^x \text{ and } (x,\tau) \in \Omega_T^n, \\ u_n(-n,\tau) = \frac{c}{r} + \frac{rL-c}{r}\, e^{-r\tau}, \quad u_n(0,\,\tau) = K, & 0 \leq \tau \leq T, \\ u_n(x,0) = \max\{L, Ke^x\}, & -n \leq x \leq 0, \end{cases} \tag{A.1}$$

where $\Omega_T^n \triangleq (-n,0)\times(0,T\,]$ and $n \in \mathbb{N}_+$ satisfying $n > \max\{\,\ln K - \ln L, \ln r + \ln K - \ln c\,\}$.

Next, we utilize the penalty method to prove the existence of the solution of (A.1). We first construct the penalty function $\beta_\varepsilon(\cdot)$ such that

$$\begin{aligned} &\beta_\varepsilon(s) \in C^\infty(\mathbb{R}), \qquad \beta_\varepsilon(s) \geq 0, \qquad \beta'_\varepsilon(s) \geq 0, \qquad \beta''_\varepsilon(s) \geq 0, \\ &\beta_\varepsilon(s) = 0 \text{ for any } s \leq -\varepsilon, \qquad \beta_\varepsilon(0) = M \triangleq qK - c > 0, \end{aligned} \tag{A.2}$$

and

$$\lim_{\varepsilon\to 0}\beta_\varepsilon(s) = \begin{cases} 0, & s < 0, \\ +\infty, & s > 0. \end{cases}$$

Then we use the following penalty problem to approximate (A.1):

$$\begin{cases} \partial_\tau u_{\varepsilon,\,n} - \mathcal{L}u_{\varepsilon,\,n} - \beta_\varepsilon(Ke^x - u_{\varepsilon,\,n}) = c & \text{in } \Omega_T^n, \\ u_{\varepsilon,\,n}(-n,\,\tau) = \frac{c}{r} + \frac{rL-c}{r}\, e^{-r\tau}, \quad u_{\varepsilon,\,n}(0,\,\tau) = K, & 0 \leq \tau \leq T, \\ u_{\varepsilon,\,n}(x,0) = \pi_\varepsilon(Ke^x - L) + L, & -n \leq x \leq 0, \end{cases} \tag{A.3}$$

where $\pi_\varepsilon(s)$ is a smoothing function for smoothing the initial value $\max\{L, Ke^x\}$, which satisfies $\pi_\varepsilon(s) \in C^\infty(\mathbb{R})$, $\pi_\varepsilon(s) \geq s$, $0 \leq \pi'_\varepsilon(s) \leq 1$, $\pi''_\varepsilon(s) \geq 0$, $\lim\limits_{\varepsilon\to 0^+}\pi_\varepsilon(s) = s^+$ and

$$\pi_\varepsilon(s) = \begin{cases} s, & s \geq \varepsilon, \\ 0, & s \leq -\varepsilon. \end{cases}$$

Lemma A.1 *For any fixed n and ε, (A.3) has a unique strong solution such that $u_{\varepsilon,n} \in W_p^{2,\,1}(\Omega_T^n) \cap C(\overline{\Omega_T^n})$ for any $1 < p < \infty$, and we have the following estimates:*

$$\max\left\{Ke^x, \frac{c}{r} + \frac{rL-c}{r}\, e^{-r\tau}\right\} \leq u_{\varepsilon,\,n} \leq K \quad \text{in } \overline{\Omega_T^n}; \tag{A.4}$$

$$0 \leq \partial_x u_{\varepsilon\, n} \leq K\left(e^x - \alpha_- e^{\alpha_-(x+n)-n}\right) \qquad \text{on } \overline{\Omega_T^n}, \tag{A.5}$$

where α_+, α_- are the positive and negative characteristic roots for the ordinary differential operator $\mathcal{L}$, respectively. That is, α_+, α_- are respectively the positive and negative roots of the following algebra equation:

$$\frac{\sigma^2}{2}\,\alpha^2 + \left(r - q - \frac{\sigma^2}{2}\right)\,\alpha - r = 0.$$

If furthermore $c \geq r\,L$, we have the following estimate:

$$\partial_\tau u_{\varepsilon\, n} \geq -r\varepsilon \;\; \text{a.e. in} \;\; \Omega_T^n. \tag{A.6}$$

Proof. The existence of the solution to (A.3) can be proved in a similar way as in [23, 25, 27], and we refer to those papers for the details. The uniqueness follows directly from the A-B-P maximum principle (see [20]).

Next, we prove (A.4). Letting $w = Ke^x$ and recalling (A.2), we calculate that

$$\begin{cases} \partial_\tau w - \mathcal{L}w - \beta_\varepsilon(Ke^x - w) - c = qKe^x - \beta_\varepsilon(0) - c = qKe^x + c - qK - c \le 0, \\ w(-n,\, \tau) = Ke^{-n} \le \min\left\{\frac{c}{r}, L\right\} \le u_{\varepsilon,\, n}(-n,\, \tau), \quad w(0,\, \tau) = K = u_{\varepsilon,\, n}(0,\, \tau), \\ w(x, 0) = Ke^x \le \max\{L, Ke^x\} \le \pi_\varepsilon(Ke^x - L) + L = u_{\varepsilon,\, n}(x, 0). \end{cases}$$

Hence, $w = Ke^x$ is a sub-solution of (A.3), and we have showed $u_{\varepsilon,\, n} \ge Ke^x$. Letting

$$w = \frac{c}{r} + \frac{rL - c}{r}\, e^{-r\tau},$$

we have that

$$\begin{cases} \partial_\tau w - \mathcal{L}w - \beta_\varepsilon(Ke^x - w) - c \le \partial_\tau w + rw - c = 0, \\ w(-n,\, \tau) = u_{\varepsilon,\, n}(-n,\, \tau), \ \ w(0,\, \tau) \le \max\left\{\frac{c}{r}, L\right\} \le \max\left\{\frac{qK}{r}, L\right\} \le K = u_{\varepsilon,\, n}(0,\, \tau), \\ w(x, 0) = L \le \max\{L, Ke^x\} \le \pi_\varepsilon(Ke^x - L) + L = u_{\varepsilon,\, n}(x, 0). \end{cases}$$

Therefore, w is another sub-solution of (A.3), and we have proved the first inequality in (A.4).

Moreover, it is easy to check that K is a super-solution of (A.3). Hence, the second inequality in (A.4) is obvious.

Next, we prove the second inequality in (A.5). Let

$$W = \frac{c}{r} + \frac{rL - c}{r}\, e^{-r\tau} + K\left(e^x - e^{\alpha_-(x+n)-n}\right).$$

If ε is small enough and n is large enough, then in the domain Ω_T^n, W satisfies

$$W(x, \tau) \ge \min\left\{\frac{c}{r}, L\right\} + Ke^x - Ke^{-n} \ge Ke^x + \varepsilon,$$

$$\text{and} \quad \begin{cases} \partial_\tau W - \mathcal{L}W + \beta_\varepsilon(Ke^x - W) - c = c + qKe^x - c > 0, \\ W(-n,\, \tau) = u_{\varepsilon,\, n}(-n,\, \tau); \quad W(0,\, \tau) \ge K + \varepsilon > K = u_{\varepsilon,\, n}(0,\, \tau), \\ W(x, 0) \ge L + Ke^x - Ke^{-n} \ge \pi_\varepsilon(Ke^x - L) + L = u_{\varepsilon,\, n}(x, 0). \end{cases}$$

Hence, W is another super-solution of (A.3), and satisfies

$$u_{\varepsilon,\, n}(x, \tau) \le \frac{c}{r} + \frac{rL - c}{r}\, e^{-r\tau} + K\left(e^x - e^{\alpha_-(x+n)-n}\right) = u_{\varepsilon,\, n}(-n, \tau) + K\left(e^x - e^{\alpha_-(x+n)-n}\right).$$

If we define

$$\overline{W}(x, \tau) = K\left(e^x - \alpha_- e^{\alpha_-(x+n)-n}\right),$$

then we have $\partial_x u_{\varepsilon,\, n}(-n,\, \tau) \le \overline{W}(-n,\, \tau)$. Since $u_{\varepsilon,\, n}(x,\, \tau) \ge Ke^x$ while $x \le 0$, and $u_{\varepsilon,\, n}(0,\, \tau) = Ke^x|_{x=0}$, we conclude that

$$\partial_x u_{\varepsilon,\, n}(0,\, \tau) \le Ke^x\Big|_{x=0} \le \overline{W}(0,\, \tau).$$

Differentiating (A.3) with respect to x, we deduce that

$$\begin{cases} (\partial_\tau - \mathcal{L})(\partial_x u_{\varepsilon,\,n} - \overline{W}) + \beta_\varepsilon'(\cdot)(\partial_x u_{\varepsilon,\,n} - \overline{W}) = -(\partial_\tau \overline{W} - \mathcal{L}\overline{W}) + \beta_\varepsilon'(\cdot)(Ke^x - \overline{W}) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \le -(\partial_\tau \overline{W} - \mathcal{L}\overline{W}) = -qKe^x < 0, \\ \partial_x u_{\varepsilon,\,n}(-n,\,\tau) - \overline{W}(-n,\,\tau) \ \le 0, \qquad \partial_x u_{\varepsilon,\,n}(0,\,\tau) - \overline{W}(0,\,\tau) \ \le 0, \\ \partial_x u_{\varepsilon,\,n}(x,0) - \overline{W}(x,\,0) \ = \pi_\varepsilon'(Ke^x - L)Ke^x - \overline{W}(x,\,0) \le Ke^x - \overline{W}(x,\,0) \le 0. \end{cases}$$

Hence, the comparison principle implies the second inequality in (A.5).

Recalling (A.4) and the boundary condition in (A.3), we deduce that for any $\tau \in [0, T]$, the following inequalities hold

$$\partial_x u_{\varepsilon,\,n}(0,\,\tau) \ge 0, \qquad \partial_x u_{\varepsilon,\,n}(-n,\,\tau) \ge 0.$$

Differentiating (A.3) with respect to x, we derive that

$$\begin{cases} (\partial_\tau - \mathcal{L})\partial_x u_{\varepsilon,\,n} + \beta_\varepsilon'(Ke^x - u_{\varepsilon,\,n})\,\partial_x u_{\varepsilon,\,n} = \beta_\varepsilon'(Ke^x - u_{\varepsilon,\,n})\,Ke^x \ge 0, \\ \partial_x u_{\varepsilon,\,n}(-n,\,\tau) \ \ge 0, \qquad \partial_x u_{\varepsilon,\,n}(0,\,\tau) \ \ge 0, \\ \partial_x u_{\varepsilon,\,n}(x,0) \ = \pi_\varepsilon'(Ke^x - L)\,Ke^x \ge 0. \end{cases}$$

Hence, the comparison principle implies the first inequality in (A.5).

In order to prove (A.6), we differentiate (A.3) with respect to τ, then we have

$$\begin{cases} (\partial_\tau - \mathcal{L})\partial_\tau u_{\varepsilon,\,n} + \beta_\varepsilon'(Ke^x - u_{\varepsilon,\,n})\,\partial_\tau u_{\varepsilon,\,n} = 0, \\ \partial_\tau u_{\varepsilon,\,n}(-n,\,\tau) \ = (c - rL)\,e^{-r\tau} \ge 0, \qquad \partial_\tau u_{\varepsilon,\,n}(0,\,\tau) \ = 0. \end{cases}$$

Recalling (A.3), we deduce that

$$\begin{aligned} &\partial_\tau u_{\varepsilon,\,n}(x,0) \ = c + \mathcal{L}u_{\varepsilon,\,n}(x,0) + \beta_\varepsilon(Ke^x - u_{\varepsilon,\,n}(x,0)) \\ \ge\ \ & c + (r-q)\,\pi_\varepsilon'(Ke^x - L)\,Ke^x - rL - r\pi_\varepsilon(Ke^x - L) + \beta_\varepsilon(Ke^x - L - \pi_\varepsilon(Ke^x - L)) \\ \ge\ \ & \begin{cases} c - rL \ge 0, & Ke^x - L < -\varepsilon, \\ c - rL - r\varepsilon \ge -r\varepsilon, & -\varepsilon \le Ke^x - L \le \varepsilon, \\ c + (r-q)\,Ke^x - rL - r(Ke^x - L) + qK - c = qK - qKe^x \ge 0, & Ke^x - L > \varepsilon. \end{cases} \end{aligned}$$

Moreover, it is clear that

$$(\partial_\tau - \mathcal{L})(-r\varepsilon) + \beta_\varepsilon'(Ke^x - u_{\varepsilon,\,n})\,(-r\varepsilon) \le -r^2\varepsilon < 0.$$

Hence, (A.6) follows from the comparison principle. ∎

Lemma A.2 *For any fixed $n \in I\!N$ satisfying $n > \max\{\ln K - \ln L,\ \ln r + \ln K - \ln c\}$, (A.1) has a unique solution $u_n \in W_p^{2,1}(\Omega_T^n \backslash B_\delta(P_0)) \cap C(\overline{\Omega_T^n})$ for any $1 < p < +\infty$, where $P_0 = (-\ln K + \ln L, 0)$, $B_\delta(P_0) = \{(x,t) : (x + \ln K - \ln L)^2 + t^2 \le \delta^2\}$. Moreover, $\partial_x u_n \in C(\Omega_T)$ and we have the following estimates:*

$$\max\left\{Ke^x, \frac{c}{r} + \frac{rL - c}{r}\,e^{-r\tau}\right\} \le u_n \le K \qquad \text{in } \overline{\Omega_T^n}; \tag{A.7}$$

$$0 \le \partial_x u_n \le K\left(e^x - \alpha e^{\alpha_-(x+n) - n}\right) \qquad \text{in } \overline{\Omega_T^n}, \tag{A.8}$$

where α_- is defined in Lemma A.1. If furthermore $c \ge r\,L$ holds, we have the following estimate:

$$\partial_\tau u_n \ge 0 \quad \text{a.e. in } \Omega_T^n. \tag{A.9}$$

Proof. From (A.2) and (A.4), we deduce that

$$0 \le \beta_\varepsilon(Ke^x - u_{\varepsilon,n}) \le \beta_\varepsilon(0) = M.$$

By employing $W_p^{2,1}$ and $C^{\alpha,\alpha/2}(0<\alpha<1)$ estimates for parabolic equations (see [16]), we derive that

$$\|u_{\varepsilon,n}\|_{W_p^{2,1}(\Omega_T^n \backslash B_\delta(P_0))} + \|u_{\varepsilon,n}\|_{C^{\alpha,\alpha/2}(\overline{\Omega_T^n})} \le C,$$

where C is a constant independent of ε. Hence, there exists a $u_n \in W_p^{2,1}(\Omega_T^n \backslash B_\delta(P_0)) \cap C(\overline{\Omega_T^n})$ and a subsequence of $\{u_{\varepsilon,n}\}$, such that as $\varepsilon \to 0^+$,

$$u_{\varepsilon,n} \rightharpoonup u_n \ \text{ in } \ W_p^{2,1}(\Omega_T^n \backslash B_\delta(P_0)) \ \text{ weakly } \ \text{ and } \ u_{\varepsilon,n} \to u_n \ \text{ in } \ C(\overline{\Omega_T^n}).$$

By applying the method in [11] or [26], we can prove that u_n is the solution of (A.1). And (A.7)-(A.9) are the consequences of (A.4)-(A.6) as $\varepsilon \to 0^+$.

Finally, we prove the uniqueness of the solution. Suppose u_n^1 and u_n^2 are two $W^{2,1}_{p,\,loc}(\Omega_T^n) \cap C(\overline{\Omega_T^n})$ solutions of (A.1) and denote

$$\mathcal{N} \triangleq \{(x,t) \in \Omega_T^n : u_n^1(x,t) < u_n^2(x,t)\}.$$

Suppose $\mathcal{N}$ is not empty, then in the domain $\mathcal{N}$,

$$Ke^x \le u_n^1(x,t) < u_n^2(x,t), \ \ \partial_t u_n^2 - \mathcal{L}u_n^2 = c, \ \ \partial_t(u_n^1 - u_n^2) - \mathcal{L}(u_n^1 - u_n^2) \ge 0.$$

Denote $W = u_n^1 - u_n^2$, we have

$$\partial_t W - \mathcal{L}W \ge 0 \ \text{ in } \mathcal{N}, \qquad W = 0 \ \text{ on } \partial_p \mathcal{N}.$$

From the A-B-P maximum principle (see [20])), we have $W \ge 0$ in $\mathcal{N}$, which contradicts the definition of $\mathcal{N}$. ■

Proof of Theorem 3.1: Rewrite (A.1) as follows:

$$\begin{cases} \partial_t u_n - \mathcal{L}u_n = f(x,t), & (x,\,t) \in \Omega_T^n, \\ u_n(-n,\,\tau) \ = \frac{c}{r} + \frac{rL-c}{r}\,e^{-r\tau}, \quad u_n(0,\,\tau) \ = K, & 0 \le \tau \le T, \\ u_n(x,0) \ = \max\{L, Ke^x\}, & -n \le x \le 0. \end{cases}$$

Since $u_n \in W^{2,1}_{p,\,\mathrm{loc}}(\Omega_T^n)$, then we have $f(x,t) \in L^p_{loc}(\Omega_T^n)$ and

$$f(x,t) = cI_{\{u_n > Ke^x\}} + qKe^x I_{\{u_n = Ke^x\}}.$$

By the $W_p^{2,\,1}$ and $C^{\alpha,\,\alpha/2}$ estimates for parabolic equations (see [16]), we deduce that for any fixed $R > \delta > 0$, the following estimates hold

$$\|u_n\|_{W_p^{2,1}(\Omega_T^R \backslash B_\delta(P_0))} \le C_{R,\delta}, \qquad \|u_n\|_{C^{\alpha,\,\alpha/2}(\overline{\Omega}_T^R)} \le C_R, \tag{A.10}$$

where $C_{R,\delta}$ depends on R and δ, C_R depends on R, but they are independent of n. Then we derive that there exists a function $u \in W^{2,1}_{p,loc}(\Omega_T) \cap C(\overline{\Omega}_T)$ and a function subsequence of $\{u_n\}$ such that for any $R > \delta > 0$, $p > 1$,

$$u_n \rightharpoonup u \ \text{ in } \ W^{2.1}_p(\Omega^R_T \backslash B_\delta(P_0)) \text{ weakly} \qquad \text{as} \quad n \to +\infty.$$

Moreover, (A.10) and the imbedding theorem imply that

$$u_n \to u \ \text{ in } \ C(\overline{\Omega}^R_T) \quad \text{and } \partial_x u_n \to \partial_x u \ \text{ in } \ C(\overline{\Omega}^R_T \backslash B_\delta(P_0)) \quad \text{as} \quad n \to +\infty. \tag{A.11}$$

By the method in [11] or [26], we can deduce that u is the strong solution of (3.5). Moreover, (A.11) implies that $\partial_x u \in C(\Omega)$. And (3.7)-(3.9) are the consequences of (A.7)-(A.9). The proof of the uniqueness is similar to the uniqueness proof in Lemma A.2. □

B The explicit solution of the PDE (3.3).

We present the explicit solution of the PDE (3.3) in this appendix. Since (3.3) is a degenerate backward problem, we make the transformation (3.4) as for the VI (3.1). Then it is not difficult to check that u is governed by

$$\begin{cases} \partial_\tau u - \mathcal{L}u = c & \text{in } \ \Omega_T, \\ u(0,\tau) \ = K, & 0 \le \tau \le T, \\ u(x,0) \ = \max\{L, Ke^x\}, & x \le 0, \end{cases} \tag{B.1}$$

It is standard to show that the classical solution of (B.1) has the following integral expression (see for example [14]):

$$\begin{aligned} u(x,\tau) \ = \ & K e^{\,x} + c \int_0^\tau \Phi_1(-x,\tau,t)\,dt - q\,K\,e^{\,x} \int_0^\tau \Phi_2(-x,\tau,t)\,dt - c\,e^{-2\alpha_1 x} \int_0^\tau \Phi_1(x,\tau,t)\,dt \\ & + q\,K\,e^{-2\alpha_1 x - x} \int_0^\tau \Phi_2(x,\tau,t)\,dt + L\,\Phi_1(\ln L - \ln K - x,\tau,0) - K\,e^{\,x}\,\Phi_2(\ln L - \ln K - x,\tau,0) \\ & - L\,e^{-2\alpha_1 x}\,\Phi_1(\ln L - \ln K + x,\tau,0) + K\,e^{-2\alpha_1 x - x}\,\Phi_2(\ln L - \ln K + x,\tau,0), \end{aligned} \tag{B.2}$$

where

$$\begin{aligned} &\Phi_1(x,\tau,t) = e^{rt - r\tau}\,\Phi(d_1(x,\tau,t)), && \Phi_2(x,\tau,t) = e^{qt - q\tau}\,\Phi(d_2(x,\tau,t)), \\ &d_1(x,\tau,t) = \frac{x}{\sigma\sqrt{\tau - t}} - \sigma\,\alpha_1\,\sqrt{\tau - t}, && d_2(x,\tau,t) = \frac{x}{\sigma\sqrt{\tau - t}} - \sigma\,(\alpha_1 + 1)\,\sqrt{\tau - t}, \\ &\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-y^2/2}\,dy, && \alpha_1 = -\frac{1}{2} + \frac{r - q}{\sigma^2}. \end{aligned}$$

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